

Pinning by holes of multiple vortices in homogenization for Ginzburg-Landau problems

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Abstract

We consider a homogenization problem for magnetic Ginzburg-Landau functional in domains with large number of small holes. For sufficiently strong magnetic field, a large number of vortices is formed and they are pinned by the holes. We establish a scaling relation between sizes of holes and the magnitude of the external magnetic field when pinned vortices are multiple and their homogenized density is described by a hierarchy of variational problems. This stands in sharp contrast with homogeneous superconductors, where all vortices are known to be simple. The proof is based on Γ -convergence approach which is applied to a coupled continuum/discrete variational problem: continuum in the induced magnetic field and discrete in the unknown finite (quantized) values of multiplicity of vortices pinned by holes.

1 Introduction

Vortices determine electromagnetic properties of superconductors that are important for practical applications (e.g., resistance). A key practical issue is to decrease the energy dissipation in superconductors, which occurs due to the motion of vortices. This dissipation can be suppressed by pinning of vortices. In particular, a physical problem of pinning in superconducting thin films with a periodic array of antidots (holes) was considered in [?] (see also references therein). This problem leads to analysis of a two-dimensional Ginzburg-Landau (GL) energy functional in a domain with periodic holes.

In this work we consider a mathematical model of pinning of vortices by many holes in relatively small superconducting samples (comparable to the

London depth) submitted to a uniform magnetic field, which is weak so that the vortices do not appear in the bulk of the superconducting sample and they may exist only in the holes.

Since modern experimental techniques allow to create very small holes, the question arises what kind of physical effects can be expected for such sample. In particular, typical experimental results lead to uniform arrays of vortices in the entire domain. In this work we present a mathematical model that leads to special “critical” scaling when a nested sequence of sub domains with vortices of increasing multiplicity appears.

We next present a brief review of relevant mathematical work. The study of pinning by a *finite number of pinning sites* was pioneered [10] where a simplified GL model (SGL) with no magnetic field was and discontinuous pinning term for a single inclusion was considered. The existence of d vortices of degree 1 inside the inclusion was established when Dirichlet boundary data with degree d . The results of [10] were subsequently generalized for the magnetic GL functional [8], [9] and pinning by a single inclusion. A comprehensive study of pinning by finitely many normal inclusions and holes for the the magnetic GL functional was performed in [1, 2]. More recently pinning by finitely many holes whose sizes goes to zero as the Ginzburg-Landau parameter goes to infinity was established in [5] for the SGL model.

Homogenization in the framework of magnetic GL model with continuous oscillating pinning term was considered in the pioneering work [3], where large number of vortices is described by the homogenized vorticity density. Since some composite superconductors are described by a discontinuous pinning term, in subsequent works [6, 7] homogenization problems for such term were address in the context of simplified GL model and special Dirichlet boundary conditions, which result in either no vortices [6] or d vortices [7] .

In this work we study a homogenization problem for a *large number of vortices* and large number of pinning holes that are described by a perforated domain Ω_ε (which corresponds to a discontinuous pinning term). This problem is described by the minimizers of the GL functional

$$GL[u, A] = \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u - iAu|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2) dx + \frac{1}{2} \int_{\Omega} (\text{curl} A - h_{ext}^\varepsilon) dx. \quad (1)$$

The unknowns here are the complex order parameter u and the vector potential of the magnetic field A , while h_{ext}^ε is given external magnetic field (positive scalar number). The domain Ω_ε in (1) is obtained by perforating a given simply connected bounded domain $\Omega \subset \mathbb{R}^2$ by a large number N_ε of small holes. Holes are all identical disks with radius ρ_ε and periodically distributed centers a_j^ε , with $\varepsilon > 0$ being a small spatial period. We assume

that

$$|\log \kappa| \gg |\log \rho_\varepsilon| \text{ and } \rho_\varepsilon \ll \varepsilon \quad (2)$$

(holes radius much greater than vortex core and much less than the spatial period ε). Moreover, we consider the following scales of magnetic field and diameters of holes,

$$h_{ext} = \lambda/\varepsilon^2, \text{ diam}(\omega_j^\varepsilon) = 2e^{-\gamma/\varepsilon^2}, \quad (3)$$

where λ and γ are fixed positive numbers.

Formal calculations show that under the scale separation condition (2) there are no vortices in the bulk of the domain Ω_ε and the term with the integrand $\frac{\kappa^2}{4}(1 - |u|^2)^2$ can be effectively replaced by the constraint $|u| = 1$. Thus we are led to the minimization problem

$$M_\varepsilon = \inf\{F_\varepsilon[u, A]; u \in H^1(\Omega_\varepsilon; S^1), A \in H^1(\Omega; \mathbb{R}^2)\} \quad (4)$$

for the functional

$$F_\varepsilon[u, A] = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u - iAu|^2 dx + \frac{1}{2} \int_{\Omega} (\text{curl} A - h_{ext}^\varepsilon)^2 dx, \quad \varepsilon > 0. \quad (5)$$

For every fixed $\varepsilon > 0$ minimizers of $(u^\varepsilon, A^\varepsilon)$ of problem (4)-(5) can be expressed in terms of degrees of the order parameter on the boundaries of holes as follows, see [2]. Let d_j^ε be (integer) degrees of u^ε on $\partial\omega_j^\varepsilon$, then the induced magnetic field $h^\varepsilon = \text{curl} A^\varepsilon$ satisfies

$$\begin{cases} -\Delta h^\varepsilon + h^\varepsilon = 0 \text{ in } \Omega_\varepsilon, \\ h^\varepsilon(x) = h_{ext}^\varepsilon \text{ on } \partial\Omega, \\ h^\varepsilon(x) = H_j^\varepsilon \text{ on } \omega_j^\varepsilon, j = 1, 2, \dots, N_\varepsilon, \\ -\int_{\partial\omega_j^\varepsilon} \frac{\partial h^\varepsilon}{\partial \nu} dx = 2\pi d_j^\varepsilon - \int_{\omega_j^\varepsilon} h(x) dx, \end{cases} \quad (6)$$

where H_j^ε are unknown constants that are part of the problem. Note that if we know the degrees d_j^ε , then the minimum (4) is given by

$$E_\varepsilon(h_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla h^\varepsilon|^2 dx + \frac{1}{2} \int_{\Omega} (h^\varepsilon - h_{ext}^\varepsilon)^2 dx, \quad (7)$$

and conversely, the minimum of (5) is obtained by minimizing (7) in integer d_j^ε , where $h^\varepsilon(x)$ defined via $\{d_j^\varepsilon\}$ as the unique solution of (6).

Our goal is to describe the asymptotic behavior of the degrees d_j^ε as $\varepsilon \rightarrow 0$.

2 Homogenization (corrector) and compactness results

We introduce rescaled quantities

$$\tilde{h}^\varepsilon = \varepsilon^2 h^\varepsilon, \quad \tilde{h}_{ext}^\varepsilon = \varepsilon^2 h_{ext}^\varepsilon, \quad \tilde{E}_\varepsilon(\tilde{h}_\varepsilon) = \varepsilon^4 E_\varepsilon(\tilde{h}_\varepsilon), \quad \tilde{M}_\varepsilon = \varepsilon^4 M_\varepsilon. \quad (8)$$

Note that h^ε (and therefore \tilde{h}_ε) is determined uniquely by the tuple of integers d_j^ε . Thus, abusing a little notations, we may write $\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) = \tilde{E}_\varepsilon(\{d_j^\varepsilon\})$. Consider a minimizing tuple of degrees $\{d_j^\varepsilon\}$, so that the corresponding solution of (6), rescaled according to (8), satisfies $\tilde{E}_\varepsilon(\tilde{h}_\varepsilon) = \tilde{M}_\varepsilon$.

First we obtain a priori bounds for these degrees d_j^ε in the following

Lemma 1. *Let d_j^ε be degrees of the minimizer of (4), then*

$$\sum (d_j^\varepsilon)^2 \leq C/\varepsilon^2, \quad (9)$$

where C is independent of ε .

Proof. The weak formulation of the problem for \tilde{h}^ε reads, find $\tilde{h}^\varepsilon \in H^1(\Omega)$ such that $\nabla \tilde{h}^\varepsilon = 0$ in all ω_j^ε and $\tilde{h}^\varepsilon = \lambda$ on $\partial\Omega$, and

$$\int_{\Omega} (\nabla \tilde{h}^\varepsilon \cdot \nabla v + \tilde{h}^\varepsilon v) dx - \varepsilon^2 \sum d_j^\varepsilon v|_{\omega_j^\varepsilon} = 0 \quad (10)$$

holds for every test function $v \in H_0^1(\Omega)$ such that $\nabla v = 0$ in all ω_j^ε . In particular, if we choose all $d_j^\varepsilon = 0$, set $v = \tilde{h}^\varepsilon - \lambda$, we get the a priori bound (for \tilde{h} corresponding to this choice of degrees d_j^ε)

$$\|\tilde{h}^\varepsilon\|_{H^1(\Omega)} := \int_{\Omega_\varepsilon} (|\nabla \tilde{h}^\varepsilon|^2 + (\tilde{h}^\varepsilon)^2) dx \leq C$$

therefore $\tilde{M}_\varepsilon \leq C$, where C is independent of ε . Hence for the minimizing tuple $\{d_j^\varepsilon\}$ we have $\tilde{E}_\varepsilon[\tilde{h}^\varepsilon] \leq C$ and thus $\|\tilde{h}^\varepsilon\|_{H^1(\Omega)} \leq C$, with another independent of ε constant C . Now choose the test function $v = \sum d_j^\varepsilon L_j^\varepsilon(x)/\log(2\rho_\varepsilon/\varepsilon)$ in (10), where

$$L_j^\varepsilon(x) = \begin{cases} \log(2|x - a_j^\varepsilon|/\varepsilon) & \text{in } B_{\varepsilon/2}(a_j^\varepsilon) \setminus B_{\rho_\varepsilon}(a_j^\varepsilon) \\ \log(2\rho_\varepsilon/\varepsilon) & \text{if } x \in B_{\rho_\varepsilon}(a_j^\varepsilon), \text{ and } 0 \text{ if } x \notin B_{\varepsilon/2}(a_j^\varepsilon). \end{cases} \quad (11)$$

Then simple computations lead to the required bound,

$$\varepsilon^2 \sum (d_j^\varepsilon)^2 = \int_{\Omega} (\nabla \tilde{h}^\varepsilon \cdot \nabla v + \tilde{h}^\varepsilon v) dx \leq \sqrt{2\pi\varepsilon^2(1 + o(\varepsilon))} \sum (d_j^\varepsilon)^2 / \gamma \|\tilde{h}^\varepsilon\|_{H^1(\Omega)},$$

$$\text{i.e. } \varepsilon^2 \sum (d_j^\varepsilon)^2 \leq 2\pi(1 + o(\varepsilon)) \|\tilde{h}^\varepsilon\|_{H^1(\Omega)}^2 / \gamma \leq C. \quad \square$$

It follows from Lemma 1 that, up to extracting a subsequence,

$$\zeta^\varepsilon = \varepsilon^2 \sum d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \rightharpoonup D(x) \text{ as distributions, and } D(x) \in L^2(\Omega). \quad (12)$$

From now on $\{d_j^\varepsilon\}$ denotes an arbitrary sequence of tuples of integers such that (9) and (12) hold, and \tilde{h} is the function associated to the tuple $\{d_j^\varepsilon\}$, i. e. $\tilde{h}^\varepsilon = \varepsilon^2 h^\varepsilon$, where h^ε is the solution of (6).

It is rather easy to see that under the above mentioned conditions we can pass to the limit in (10) to get that \tilde{h}^ε converges H^1 -weakly to the solution \bar{h} of the homogenized problem

$$\begin{cases} -\Delta \bar{h} + \bar{h} = 2\pi D(x) & \text{in } \Omega \\ \bar{h} = \lambda & \text{on } \partial\Omega. \end{cases} \quad (13)$$

(for details see Lemma 3 below). However this result is too weak for our principal goal of describing the limiting vorticity $D(x)$, which will be (naturally) done by calculating the Γ -limit of the functionals \tilde{E}^ε . In the next step we introduce a corrector and get the strong H^1 -convergence.

We consider the ansatz,

$$\tilde{h}^\varepsilon(x) = \bar{h}^\varepsilon(x) - \varepsilon^2 \sum d_j^\varepsilon L_j^\varepsilon(x) = \bar{h}^\varepsilon(x) + R^\varepsilon, \quad (14)$$

where functions $L_j^\varepsilon(x)$ are given by (11). The problem for \bar{h}^ε (in its weak form) is, find $\bar{h}^\varepsilon \in H^1(\Omega)$ such that $\nabla \bar{h}^\varepsilon = 0$ in all ω_j^ε and $\bar{h}^\varepsilon = \lambda$ on $\partial\Omega$, and

$$\begin{aligned} \int_{\Omega} (\nabla \bar{h}^\varepsilon \cdot \nabla v + \bar{h}_\varepsilon v) dx + \sum \int_{B_{\varepsilon/2}(a_j^\varepsilon)} v R^\varepsilon(x) dx \\ = \varepsilon \sum d_j^\varepsilon \int_{\partial B_\varepsilon(a_j^\varepsilon)} v ds \end{aligned} \quad (15)$$

holds for every test function $v \in H_0^1(\Omega)$ such that $\nabla v = 0$ in all ω_j^ε .

Lemma 2. *Under condition (12) functions \bar{h}^ε converge H^1 -strongly to \bar{h} , the unique solution of (13).*

Remark 1. *Lemma 2 shows that the function $R^\varepsilon(x) = -\varepsilon^2 \sum d_j^\varepsilon L_j^\varepsilon(x)$ is a corrector, so that $\bar{h}^\varepsilon = \tilde{h}^\varepsilon - R^\varepsilon$ converges strongly in $H^1(\Omega)$.*

Proof. Using Lemma 1 one shows that R^ε converges H^1 -weakly to zero, and therefore $\bar{h}^\varepsilon \rightharpoonup \bar{h}$, up to extracting a subsequence. To prove that \bar{h} solves (13) we consider test functions $v^\varepsilon = v(x) + \sum \phi(x/\rho_\varepsilon)(v(a_j^\varepsilon) - v(x))$, where

$v \in C_0^\infty(\Omega)$ is an arbitrary function, and $\phi(x)$ is smooth cut-off function such that $\phi = 1$ if $|x| \leq 1$ and $\phi = 0$ if $|x| > 2$. Set $v = v^\varepsilon$ in (15) and pass to the limit as $\varepsilon \rightarrow 0$ to get

$$\int_{\Omega} (\nabla \bar{h} \cdot \nabla v + \bar{h}v) dx = 2\pi \int_{\Omega} D(x)v dx.$$

Thus \bar{h} solves (13).

Next we show that \bar{h}^ε converges H^1 -strongly to \bar{h} . We set $v = \bar{h}^\varepsilon - \lambda$ to obtain in the limit $\varepsilon \rightarrow 0$,

$$\limsup \int_{\Omega} \nabla \bar{h}^\varepsilon \cdot \nabla \bar{h}^\varepsilon dx = \limsup \varepsilon \sum d_j^\varepsilon \int_{\partial B_\varepsilon(a_j^\varepsilon)} (\bar{h}^\varepsilon - \lambda) ds + \int_{\Omega} (\lambda - \bar{h})\bar{h} dx.$$

By the Pioncaré inequality, we have

$$\frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(a_j^\varepsilon)} \bar{h}^\varepsilon ds = \frac{1}{\varepsilon^2} \int_{\Pi_j^\varepsilon} \bar{h}^\varepsilon dx + O(1) \left(\int_{\Pi_j^\varepsilon} |\nabla \bar{h}^\varepsilon|^2 dx \right)^{1/2},$$

where Π_j^ε is the cell with the center at a_j^ε and the side length ε . Therefore

$$\lim \varepsilon \sum d_j^\varepsilon \int_{\partial B_\varepsilon(a_j^\varepsilon)} \bar{h}^\varepsilon ds = 2\pi \int_{\Omega} D(x)\bar{h} dx,$$

where we have used Lemma 1, (12) and the fact that $\bar{h}^\varepsilon \rightarrow \bar{h}$ strongly in $L^2(\Omega)$. Thus, taking into account (13), we finally get

$$\limsup \int_{\Omega} \nabla \bar{h}^\varepsilon \cdot \nabla \bar{h}^\varepsilon dx = 2\pi \int_{\Omega} D(x)(\bar{h} - \lambda) dx + \int_{\Omega} (\lambda - \bar{h})\bar{h} dx = \int_{\Omega} \nabla \bar{h} \cdot \nabla \bar{h} dx.$$

This implies that $\bar{h}^\varepsilon \rightarrow \bar{h}$ strongly in $H^1(\Omega)$. \square

As a corollary of Lemma 2 we obtain

Lemma 3. *Under condition (12) the following energy expansion holds,*

$$\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) = \bar{E}_1(\bar{h}) + \pi\gamma\varepsilon^2 \sum (d_j^\varepsilon)^2 + o(1), \quad (16)$$

where

$$\bar{E}_1(\bar{h}) = \frac{1}{2} \int_{\Omega} |\nabla \bar{h}|^2 dx + \frac{1}{2} \int_{\Omega} (\bar{h} - \lambda)^2 dx \quad (17)$$

Proof. Since $\bar{h}^\varepsilon \rightarrow \bar{h}$ strongly in $H^1(\Omega)$ while $R^\varepsilon \rightarrow 0$ weakly in $H^1(\Omega)$, we have

$$\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) = \bar{E}_1(\bar{h}^\varepsilon) + \frac{1}{2} \int_{\Omega} |\nabla R^\varepsilon|^2 dx + o(1) = \bar{E}_1(\bar{h}) + \frac{1}{2} \int_{\Omega} |\nabla R^\varepsilon|^2 dx + o(1)$$

The straightforward calculation of the second term in this expansion yields (16). \square

3 Limiting vorticity via Γ -convergence

The main result of this work describing the limiting vorticity is obtained by proving the Γ -convergence of functionals \tilde{E}_ε with respect to weak convergence (12) of vorticity measures,

$$\tilde{E}_\varepsilon \text{ } \Gamma\text{-converge to } \bar{E}_0(D) = \bar{E}_1(\bar{h}) + \pi\gamma \int_{\Omega} \Phi(D(x))dx \text{ as } \varepsilon \rightarrow 0, \quad (18)$$

where \bar{h} is the unique solution of (13). More precisely we demonstrate that

(i) ($\Gamma - \liminf$ inequality) if conditions (9) and (12) are satisfied then

$$\liminf \tilde{E}_\varepsilon(\{d_j^\varepsilon\}) \geq \bar{E}_0(D); \quad (19)$$

(ii) ($\Gamma - \limsup$ inequality) $\forall D(x) \in L^2(\Omega)$ there is a (recovery) sequence of tuples $\{d_j^\varepsilon\}$ satisfying conditions (9) and (12) and such that

$$\limsup \tilde{E}_\varepsilon(\{d_j^\varepsilon\}) \leq \bar{E}_0(D). \quad (20)$$

The function Φ in the limit functional (18) is a continuous piecewise linear function such that $\Phi(d) = d^2$ at integer points d . It describes the homogenized density of energies of individual vortices, whereas \bar{E}_1 corresponds to the interaction a vortex with magnetic field due to other vortices and external field.

Thanks to the energy expansion (16), for "lim inf" inequality we need only to prove the lower bound

$$\liminf \varepsilon^2 \sum (d_j^\varepsilon)^2 \geq \pi\gamma \int_{\Omega} \Phi(D(x)) dx. \quad (21)$$

Since the left hand side of (21) is nonlinear (quadratic) function of d_j^ε , we use an analog of Young measures (see Appendix).

3.1 Lower bound

Spread the measure ζ^ε , which is the sum of point masses, over periodicity cells Π_j^ε by setting $D^\varepsilon = d_j^\varepsilon$ in Π_j^ε (Π_j^ε is the cell centered at a_j^ε). Then represent D^ε as

$$D^\varepsilon(x) = \sum_{k \in \mathbb{Z}} k \mu_k^\varepsilon(x), \text{ where } \mu_k^\varepsilon(x) = \begin{cases} 1 & \text{in } \Pi_j^\varepsilon, \text{ if } k = d_j^\varepsilon \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

We extend D^ε and μ_k^ε , $k \neq 0$, on Ω by setting $D^\varepsilon = \mu_k^\varepsilon = 0$ in $\Omega \setminus \cup \Pi_j^\varepsilon$ and also set $\mu_0 = 1$ in $\Omega \setminus \cup \Pi_j^\varepsilon$. The functions $\mu_k^\varepsilon(x)$ satisfy $\mu_k^\varepsilon \geq 0$ and $\sum \mu_k^\varepsilon = 1$ and therefore form a partition of unity. Clearly, we can extract a subsequence such that

$$\mu_k^\varepsilon \rightharpoonup \mu_k \text{ weakly in } L^2(\Omega) \forall k \in \mathbb{Z}. \quad (23)$$

Thanks to the bound (9) we have

$$\sum_{k \in \mathbb{Z}} k^2 \int_{\Omega} \mu_k^\varepsilon(x) = \varepsilon^2 \sum (d_j^\varepsilon)^2 \leq C.$$

Hence the limit functions μ_k also form a partition of unity,

$$\mu_k \geq 0 \text{ and } \sum \mu_k = 1. \quad (24)$$

Moreover, the function $D(x)$ defined in (12) admits the representation

$$D(x) = \sum k \mu_k(x) \quad (25)$$

and it is easy to see that the following inequality holds,

$$\liminf \varepsilon^2 \sum (d_j^\varepsilon)^2 \geq \sum_{k \in \mathbb{Z}} k^2 \int_{\Omega} \mu_k(x). \quad (26)$$

In order to obtain a lower bound in terms of $D(x)$ we prove the following simple

Lemma 4. *Given D (real number), then*

$$\Phi(D) = \min \left\{ \sum_{k \in \mathbb{Z}} k^2 \mu_k; \mu_k \geq 0, \sum \mu_k = 1, \sum k \mu_k = D \right\} \quad (27)$$

is $\Phi(D) = (2k+1)|D| - k - k^2$ if $k \leq |D| < k+1$, $k = 0, 1, 2, \dots$. Moreover, if $D = d$ is an integer then the unique minimizing tuple is $m_d = 1$, $m_k = 0 \forall k \neq d$. In the case D is non integer, represent D as the convex hull of two nearest integers d and $d+1$, $D = \alpha d + (1-\alpha)(d+1)$, then the unique minimizing tuple is $\mu_d = \alpha$, $\mu_{d+1} = 1-\alpha$, $\mu_k = 0 \forall k \notin \{d, d+1\}$.

Proof. If $D = d$ is an integer then, clearly, $\Phi(D) \leq d^2$, and by Cauchy-Schwarz inequality we have

$$d^2 = (\sum k \mu_k)^2 \leq \sum k^2 \mu_k = \Phi(D).$$

Thus $\Phi(D) = d^2$ and the minimizing tuple is $m_d = 1, m_k = 0 \forall k \neq d$.

Consider now the case when D is non integer and $D = \alpha d + (1 - \alpha)(d + 1)$. Let $\{\mu_k\}$ be a minimizing tuple. If $\mu_k > 0$ for some $k < d$ then there is $\mu_l > 0$ for some $l \geq d + 1$. Decrease μ_k and μ_l by a sufficiently small $\delta > 0$ and increase μ_{k+1} and μ_{l-1} by δ . This modification change neither (24) nor (25) but decrease the value of functional $\sum k^2 \mu_k$. Therefore $\mu_k = 0 \forall k \notin \{d, d + 1\}$. The case when $\mu_k > 0$ for some $k > d + 1$ is similar. It follows from (24) and (25) that $\mu_d = \alpha, \mu_{d+1} = 1 - \alpha$ and straightforward calculations yield the result. \square

3.2 Upper bound

In order to complete the proof of Γ -convergence (18) we have to show the lim sup-inequality, i.e., given $D \in L^2(\Omega)$, we need to construct a sequence of tuples $\{d_j^\varepsilon\}$ satisfying the boundedness condition (9), that converge to $D(x)$ in the sense of (12) and satisfy inequality (20).

The limiting functional $\bar{E}_0(D)$ is continuous with respect to the strong convergence in $L^2(\Omega)$ therefore it is sufficient to establish the lim sup-inequality for $D \in C_0^\infty(\Omega)$ and then use density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$.

Note that we not only need the converge of tuples in the sense of (12) but more importantly we need the convergence of energies which does not follow from (12). The key issue in the construction of the upper bound is that different configurations of vortices may lead tho the same vorticity $D(x) = \sum k \mu_k(x)$, however these configurations can be distinguished by $\sum k^2 \mu_k(x)$ (which is equal to $\Phi(D)$ for optimal μ_k given by Lemma 4). Thus we need to chose d_j^ε that define μ_k^ε via (22) so that the limiting values μ_k are optimal in the sense of (27). More precisely, if we represent $D(x)$ for fixed $x \in \Omega$ as a convex hull of the nearest integers d and $d + 1$, $D(x) = \alpha d + (1 - \alpha)(d + 1)$, we must have only holes with degree d and $d + 1$ in a small neighborhood of x and in this neighborhood $\#\{\text{holes with degree } d\} / \#\{\text{holes with degree } d + 1\}$ should be approximately equal to $\alpha / (\alpha + 1)$.

Recall that the centers of holes a_j^ε form ε -lattice and therefore partition the domain Ω into fine squares Π_j^ε of side length ε . We choose fixed sufficiently large integer M and consider squares K_k with side length $(2M + 1)\varepsilon$ and centers at points a_k^ε which form an $(2M + 1)\varepsilon$ -periodic lattice. Consider a square K_k strictly included in Ω (it contains exactly $(2N + 1)^2$ holes). Let D_k denote the mean value of $D(x)$ on K_k ,

$$D_k = \frac{1}{(2M + 1)^2 \varepsilon^2} \int_{K_k} D(x) dx,$$

and let d_k be the integer such that $d_k \leq D_k < d_k + 1$. Represent D_k as the

convex combination of d_k and d_k+1 , $D_k = \alpha_k d_k + (1-\alpha_k)(d_k+1)$ ($0 \leq \alpha_k < 1$). Next we chose the largest integer $R > 0$ such that $R/(2M+1)^2 \leq \alpha_k$ and set $d_j^\varepsilon := d_k$ for R holes ω_j^ε in K_k and $d_j^\varepsilon := d_k + 1$ for other holes in K_k . We repeat this procedure for all squares K_k lying strictly inside Ω and set degrees of remaining holes to be zero. Let functions μ_k^ε be defined by (22), then we have

$$\varepsilon^2 \sum (d_j^\varepsilon)^2 = \sum_{l \in \mathbb{Z}} l^2 \int_{\Omega} \mu_l^\varepsilon(x) dx.$$

We claim that for sufficiently small $\varepsilon > 0$

$$\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) \leq \bar{E}_0(D) + \delta_M, \quad (28)$$

where $\delta_M \rightarrow 0$ as $M \rightarrow \infty$. Indeed, up to extracting a subsequence, $\mu_l^\varepsilon \rightarrow \mu_l(x)$ weakly in $L^2(\Omega)$ for all $l \in \mathbb{Z}$. The limiting vorticity $\tilde{D}(x)$ (which depends on M) is given by $\tilde{D}(x) = \sum l \mu_l(x)$. Due to the construction of tuples $\{d_j^\varepsilon\}$ we have, $|\mu_d(x) - \alpha| \leq 1/(2M+1)^2$ and $|\mu_{d+1}(x) - (1-\alpha)| \leq 1/(2M+1)^2$ for every point $x \in \Omega$ such that $\text{dist}(D(x), \mathbb{Z}) \geq 1/(2M)^2$, where $\alpha d + (1-\alpha)(d+1) = D(x)$ is the representation of $D(x)$ as the convex combination of nearest integers. On the other hand $|\mu_d(x) - 1| \leq 2/(2M)^2$ if $x \in \Omega$ and $|D(x) - d| \leq 1/(2M)^2$ for some integer d . It follows that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \sum (d_j^\varepsilon)^2 = \sum l^2 \int_{\Omega} \mu_l(x) \leq \int_{\Omega} \Phi(\tilde{D}(x)) dx + C/M^2.$$

We also have $|\tilde{D}(x) - D(x)| \leq C/M^2$ in Ω . Thus, by virtue of Lemma 3 we obtain (28). Next choosing a suitable sequence of increasing integers $M = M_\varepsilon$ we get the required upper bound.

3.3 Γ -convergence theorem

We summarize results of this Section in the following

Theorem 1. *The functionals \tilde{E}_ε Γ -converge to $\bar{E}_0(D)$ as $\varepsilon \rightarrow 0$. The limit functional $E_0(D)$ is given by*

$$E_0(D) = \frac{1}{2} \int_{\Omega} (|\nabla \bar{h}|^2 + (\bar{h} - \lambda)^2) dx + \pi\gamma \int_{\Omega} \Phi(D(x)) dx, \quad (29)$$

where $\bar{h} = \bar{h}(D)$ is the unique solution of (13) and $\Phi(D) = (2k+1)|D| - k - k^2$ if $k \leq |D| < k+1$, $k = 0, 1, 2, \dots$, see Fig 1.

This yields the main homogenization result of this work

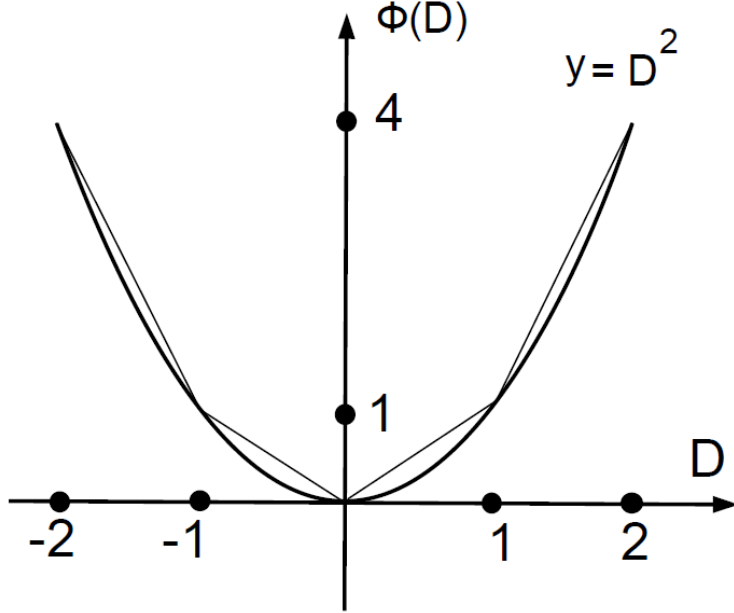


Figure 1: Function $\Phi(D)$

Theorem 2 (homogenized vorticity). *Let $\{d_j^\varepsilon\}$ be a tuple of integer degrees solving the minimization problem (4), (5) (i.e. the solution h^ε of problem (6) minimizes the functional (7)), then*

$$\varepsilon^2 \sum d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \rightharpoonup D(x), \text{ in the sense of distributions,} \quad (30)$$

where D is the unique minimizer of the functional $E_0(D)$ in $L^2(\Omega)$, $E_0(D)$ being given by (29).

Proof. Note that the Γ -limit functional $E_0(D)$ is strictly convex continuous and coercive, therefore it has the unique minimizer. On the other hand, by Lemma 1 the weak limit $\varepsilon^2 \sum d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \rightharpoonup D(x)$ exists (up to extracting a subsequence) and $D \in L^2(\Omega)$. Therefore, due to the classical properties of Γ -convergence, D is the unique minimizer of $E_0(D)$. \square

4 Analysis of the limit problem via convex duality. Hierarchy of multiplicities

We use convex duality (see, e.g., [12]) to pass from problem

$$M_\lambda = \min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla(h - \lambda)|^2 + (h - \lambda)^2 + 2\pi\gamma\Phi((- \Delta h + h)/(2\pi))) dx; \right. \\ \left. (h - \lambda) \in H_0^1(\Omega), -\Delta h + h \in L^2(\Omega) \right\} \quad (31)$$

to the dual one

$$-M_\lambda = \min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2) dx + \mathcal{F}^*(-f); f \in H_0^1(\Omega) \right\}, \quad (32)$$

where $\mathcal{F}^*(f)$ is the Legendre transform of the functional

$$\mathcal{F}(\kappa) = \pi\gamma \int_{\Omega} \Phi((- \Delta \kappa + \kappa + \lambda)/(2\pi)),$$

i.e.

$$\mathcal{F}^*(f) = \sup \left\{ \int_{\Omega} (\nabla f \cdot \nabla \kappa + f\kappa) dx + \mathcal{F}(\kappa); \kappa \in H_0^1(\Omega) \right\}.$$

Due to the fact that $\mathcal{F}(\kappa)$ is lower semicontinuous, the minimizer $(\bar{h} - \lambda)$ of (31) and minimizer \bar{f} of (32) coincide (moreover M_λ in (31) and (32) is the same). Thus, we have

$$2\pi D(x) = -\Delta \bar{h} + \bar{h} = -\Delta \bar{f} + \bar{f} + \lambda \quad (33)$$

The calculation of the Legendre transform $\mathcal{F}^*(f)$ is reduced to the calculation of the Legendre transform Φ^* of the function $\pi\gamma\Phi(\kappa/(2\pi))$. Indeed, if we use integration by parts we derive

$$\int_{\Omega} (\nabla f \cdot \nabla \kappa + f\kappa) dx + \mathcal{F}(\kappa) = \\ \int_{\Omega} (-\Delta \kappa + \kappa + \lambda)f dx + \int_{\Omega} (\pi\gamma\Phi((- \Delta \kappa + \kappa + \lambda)/(2\pi)) - \lambda f) dx,$$

and therefore

$$\mathcal{F}^*(-f) = \int_{\Omega} (\Phi^*(-f) + \lambda f) dx. \quad (34)$$

The Legendre transform $\Phi^*(f)$ of $\pi\gamma\Phi(\kappa/(2\pi))$ is given by $\Phi^*(f) = 0$ for $|f| \leq \gamma/2$ and $\Phi^*(f) = 2\pi k|f| - \pi\gamma k^2$ for $k\gamma - \gamma/2 \leq |k| \leq k\gamma + \gamma/2$.

Thus (31) is equivalent to the problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 2\Phi^*(f) + 2\lambda f) dx; f \in H_0^1(\Omega) \right\}, \quad (35)$$

and the limit vorticity is defined in terms of the minimizer \bar{f} by the formula (33).

An important role in the analysis of problem (35) plays the pointwise monotonicity of minimizers in λ shown in the following

Lemma 5. *Let \bar{f}_λ be the minimizer of (35), then (i) $\bar{f}_\lambda \leq 0$ in Ω for every $\lambda > 0$, and $\bar{f}_\alpha \leq \bar{f}_\beta$ in Ω if $\alpha > \beta$.*

Proof. Approximate the function $\Phi^*(f)$ by smooth convex functions $\tilde{\Phi}_\delta^*(f)$, where $\delta > 0$ is a small parameter. Set

$$\tilde{\Phi}_\delta^*(f) = \frac{1}{2\delta} \int_{f-\delta}^{f+\delta} \Phi^*(z) dz$$

Clearly $\tilde{\Phi}_\delta^* \in C^1(\mathbb{R})$ and $\tilde{\Phi}_\delta^*$ is convex thanks to the convexity of $\Phi^*(f)$. Let the \tilde{f}_λ be the minimizer of the functional (35) with $\tilde{\Phi}_\delta^*$ in place of Φ^* . It is obvious that this minimizer is continuous function. Assume that the function $\tilde{f} = \tilde{f}_\beta - \tilde{f}_\alpha$ has a negative minimum. Subtracting Euler-Lagrange equation for \tilde{f}_β from that for \tilde{f}_α we obtain

$$-\Delta \tilde{f} + \tilde{f} + (\beta - \alpha) + (\tilde{\Phi}_\delta^*)'(f_\beta) - (\tilde{\Phi}_\delta^*)'(f_\alpha) = 0$$

At the minimum point of \tilde{f} we have that $-\Delta \tilde{f} \leq 0$, $\tilde{f} < 0$, $\beta - \alpha < 0$ and $(\tilde{\Phi}_\delta^*)'(\tilde{f}_\beta) - (\tilde{\Phi}_\delta^*)'(\tilde{f}_\alpha) \leq 0$. Thus we have a contradiction and therefore $\tilde{f}_\beta \geq \tilde{f}_\alpha$. In particular, setting $\beta = 0$ we get $\tilde{f}_\lambda \leq 0$ for $\lambda > 0$.

The result follows by passing to the limit $\delta \rightarrow 0$. \square

4.1 Weak magnetic fields: zero vorticity

Let us consider weak magnetic fields $h_{ext}^\varepsilon = \lambda/\varepsilon^2$, such that $\lambda > 0$ is small. It is natural to expect that for such magnetic fields the minimizer \bar{f}_λ of the problem (35) satisfies $-\gamma/2 < \bar{f}_\lambda \leq 0$ therefore $\Phi^*(\bar{f}_\lambda + v) = 0$ in Ω for every sufficiently small smooth test function, and this implies that \bar{f}_λ must solve the problem

$$\begin{cases} \Delta f = f + \lambda & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \quad (36)$$

More precisely, the case of zero vorticity is described by

Proposition 1. *Let f_1 be the solution of (36) for $\lambda = 1$ and γ is defined in (3). If*

$$\lambda \leq \lambda_{\text{cr1}} := \frac{\gamma}{2 \max |f_1|} \quad (37)$$

then the minimizer \bar{f}_λ of (35) is given by $\bar{f}_\lambda = \lambda f_1$, and, according to (33), $D(x) = 0$.

Proof. Since

$$\frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 2\Phi^*(f) + 2\lambda f) dx \geq \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 2\lambda f) dx \quad (38)$$

for every f , and (38) becomes equality if f is the minimizer of the right hand side, $f = \lambda f_1$, the result follows. \square

4.2 Moderate magnetic fields: simple vortices

When the parameter λ increases λf_1 is no longer the minimizer of (35), the latter is rather described by the variational problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 4\pi(|f| - \gamma/2)_+ + 2\lambda f) dx; f \in H_0^1(\Omega) \right\}. \quad (39)$$

Proposition 2. *Let g_λ be the minimizer of (39) and let*

$$\lambda_{\text{cr2}} := \max\{\lambda > 0; \max |g_\lambda| \leq 3\gamma/2\}, \quad (40)$$

then the minimizer \bar{f}_λ of (35) coincides with that of (39).

Proof. The proof is identical to that of Proposition 1. Note that we make use here of Lemma 5 to get that the minimizer g_λ of (39) satisfies the inequality $g_\lambda \geq -3\gamma/2$ in Ω when $\lambda \leq \lambda_{\text{cr2}}$. \square

In order to describe the limiting vorticity function $D(x)$ we have to study in more details problem (39).

Proposition 3. *If $\lambda_{\text{cr1}} < \lambda \leq 2\pi + \gamma/2$ then (39) is reduced to the obstacle problem*

$$\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 2\lambda f) dx; f \in H_0^1(\Omega), |f| \leq \gamma/2 \right\}. \quad (41)$$

The minimizer \bar{f}_λ takes value $-\gamma/2$ on a set with nonzero measure ($2D$ Lebesgue). Moreover, the vorticity $D(x)$ is zero in the domain where $\bar{f}_\lambda(x) > -\gamma/2$ and $D(x) = (\lambda - \gamma/2)/(2\pi)$ otherwise.

If $\max\{\lambda_{\text{cr1}}, 2\pi + \gamma/2\} < \lambda \leq \lambda_{\text{cr2}}$ then $D(x) = 1$ when $\bar{f}_\lambda(x) < -\gamma/2$ and $D(x) = 0$ if $\bar{f}_\lambda(x) \geq -\gamma/2$, where \bar{f}_λ is the minimizer of (39).

Remark 2. It follows from Proposition 3 that the vorticity $D(x)$ is zero in a subdomain of Ω where $\bar{f}_\lambda > -\gamma/2$ and $0 < D(x) \leq 1$ when $\bar{f}_\lambda \leq -\gamma/2$. Moreover, there are two scenarios, if $\lambda_{\text{cr1}} < \lambda \leq 2\pi + \gamma/2$ then $0 < D(x) < 1$ in the set where $\bar{f}_\lambda(x) \leq -\gamma/2$ (holes with degrees one and zero), and if $\max\{\lambda_{\text{cr1}}, 2\pi + \gamma/2\} < \lambda \leq \lambda_{\text{cr2}}$ then $D(x) = 1$ when $\bar{f}_\lambda(x) \leq -\gamma/2$ (all holes in this set have degree one).

Proof. Let $\lambda_{\text{cr1}} < \lambda \leq 2\pi + \gamma/2$ then $f^2 + 4\pi(|f| - \gamma/2)_+ + 2\lambda f > \gamma^2/4 - \lambda\gamma$ when $f < -\gamma/2$. It follows that the minimizer \bar{f}_λ of (39) satisfies the pointwise inequality $\bar{f}_\lambda \geq -\gamma/2$. Clearly we also have $\bar{f}_\lambda \leq 0$. Thus \bar{f}_λ minimizes (44). If we assume that \bar{f}_λ satisfies the strict inequality $\bar{f}_\lambda(x) > -\gamma/2$ for a.e. $x \in \Omega$, then $-\Delta\bar{f}_\lambda(x) + \bar{f}_\lambda(x) + \lambda = 0$ for a.e. $x \in \Omega$ and therefore \bar{f}_λ is the solution of problem (36). Since $\lambda > \lambda_{\text{cr1}}$ we have a contradiction with the pointwise bound $\bar{f}_\lambda \geq -\gamma/2$.

In the case when $\max\{\lambda_{\text{cr1}}, \pi + \gamma/2\} < \lambda \leq \lambda_{\text{cr2}}$ we clearly have $-\Delta\bar{f}_\lambda(x) + \bar{f}_\lambda(x) + \lambda = 0$ when $-\gamma/2 < \bar{f}_\lambda \leq 0$, and $-\Delta\bar{f}_\lambda(x) + \bar{f}_\lambda(x) + \lambda = 2\pi$ when $\bar{f}_\lambda < -\gamma/2$. Thus we only need to show that the level set $\bar{f}_\lambda = -\gamma/2$ has zero measure. To this end consider the set $W = \{x \in \Omega; -\gamma/2 \geq \bar{f}_\lambda > -\gamma/2 - \delta\}$, where $\delta > 0$. For sufficiently small δ the boundary of W can be divided into two nonempty parts $S_1 = \{x \in \partial W; \bar{f}_\lambda = -\gamma/2\}$ and $S_2 = \{x \in \partial W; \bar{f}_\lambda = -\gamma/2 - \delta\}$. Both sets S_1 and S_2 have zero measure. Consider the function U such that $\Delta U = 0$ in the interior of W , $U = -\gamma/2$ on S_1 , and $U = -\gamma/2 - \delta$ on S_2 . By the maximum principle $U < -\gamma/2$ in the interior of W . On the other hand $\bar{f}_\lambda \leq U$ (otherwise $\min\{\bar{f}_\lambda, U\}$ is a minimizer). Thus the level set $\bar{f}_\lambda = -\gamma/2$ coincides with S_1 and has zero measure. \square

4.3 Stronger magnetic fields: multiple vortices

For $\lambda > \lambda_{\text{cr2}}$ vortices with multiplicity two appear. Similarly to the case of simple vortices there are two scenarios depending on whether $\lambda_{\text{cr2}} < \gamma/2 + 2\pi$ or $\lambda_{\text{cr2}} \geq \gamma/2 + 2\pi$. Define

$$\lambda_{\text{cr3}} := \max\{\lambda > 0; \max|g_\lambda| \leq 5\gamma/2\}, \quad (42)$$

where g_λ is the minimizer of the problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 4\pi((|f| - \gamma/2)_+ + (|f| - 3\gamma/2)_+) + 2\lambda f) dx; \right. \\ \left. f \in H_0^1(\Omega) \right\}. \quad (43)$$

Proposition 4. *If $\lambda_{\text{cr}2} < \lambda \leq 4\pi + 3\gamma/2$ then (43) is reduced to the obstacle problem*

$$\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + f^2 + 4\pi(|f| - \gamma/2)_+ + 2\lambda f) dx; f \in H_0^1(\Omega), |f| \leq 3\gamma/2 \right\}. \quad (44)$$

The minimizer \bar{f}_{λ} takes value $-3\gamma/2$ on a set with nonzero measure (2D Lebesgue). Moreover, the vorticity $D(x)$ is zero in the domain where $\bar{f}_{\lambda}(x) > -\gamma/2$, $D(x) = 1$ when $-3\gamma/2 < \bar{f}_{\lambda}(x) < -\gamma/2$ and $D(x) = (\lambda - 3\gamma/2)/(2\pi)$ when $\bar{f}_{\lambda}(x) = -3\gamma/2$.

If $\max\{\lambda_{\text{cr}2}, 4\pi + 3\gamma/2\} < \lambda \leq \lambda_{\text{cr}3}$ then $D(x) = 0$ when $\bar{f}_{\lambda}(x) > -\gamma/2$, $D(x) = 1$ if $-3\gamma/2 < \bar{f}_{\lambda}(x) < -\gamma/2$ and $D(x) = 2$ when $\bar{f}_{\lambda}(x) = -3\gamma/2$, where \bar{f}_{λ} is the minimizer of (43).

Remark 3. *In the case when $\lambda_{\text{cr}2} < \lambda \leq 4\pi + 3\gamma/2$ we see that vortices with multiplicities one and two coexist in the set where $\bar{f}_{\lambda}(x) = -3\gamma/2$, while all holes have degrees one in the domain where $-3\gamma/2 < \bar{f}_{\lambda}(x) < -\gamma/2$ and zero degrees in the domain where $\bar{f}_{\lambda}(x) > -\gamma/2$. If $\max\{\lambda_{\text{cr}2}, 4\pi + 3\gamma/2\} < \lambda \leq \lambda_{\text{cr}3}$ there are three subdomains, where $\bar{f}_{\lambda}(x) > -\gamma/2$, $-3\gamma/2 < \bar{f}_{\lambda}(x) < -\gamma/2$ and $\bar{f}_{\lambda}(x) = -3\gamma/2$. All holes in these domains have degrees zero, one and two, correspondingly.*

The proof of this result is similar to the previous ones. Further increase of the magnetic field leads to vortices with higher multiplicities in nested subdomains.

Proposition 5. *There exists a strictly increasing sequence of critical values $\lambda_{\text{cr}j}$, $j = 1, 2, \dots$ such that if $\lambda_{\text{cr}j} < \lambda < \lambda_{\text{cr}(j+1)}$ the limiting vorticity takes constant values in subsets $\Omega_k \setminus \Omega_{k+1}$, where $\Omega_k = \Omega_k(\lambda)$, $k = 0, 1, \dots, j$ are strictly nested sets and $\Omega_0 = \Omega$. Namely, the vorticity $D(x) = 0$ in $\Omega_0 \setminus \Omega_1$ and $D(x) = k$ in $\Omega_k \setminus \Omega_{k+1}$, $k \leq j - 1$. Finally, when $x \in \Omega_j$ then there are two scenarios: (i) if $\lambda < 2\pi j + (j - 1/2)\gamma$ then $(j - 1) < D(x) < j$ otherwise (ii) $D(x) = j$.*

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References

- [1] S. Alama, L. Bronsard, Pinning effects and their breakdown for a Ginzburg-Landau model with normal inclusions. *J. Math. Phys.* 46 (2005), no. 9, 39 pp.
- [2] S. Alama, L. Bronsard, Vortices and pinning effects for the Ginzburg-Landau model in multiply connected domains. *Comm. Pure Appl. Math.* 59 (2006), no. 1, 36 - 70.
- [3] A. Aftalion, E. Sandier, S. Serfaty, Pinning phenomena in the Ginzburg-Landau model of superconductivity. *J. Math. Pures Appl.* (9) 80 (2001), no. 3, 339 - 372.
- [4] E.J. Balder, Lectures on Young Measures, *Cah. de Ceremade* (1995).
- [5] M. Dos Santos and O. Misiats, "Ginzburg-Landau model with small pinning domains." *Networks and Heterogeneous Media*, submitted.
- [6] M. Dos Santos, P. Mironescu and O. Misiats, "The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I : the zero degree case", *Communications in Contemporary Mathematics*, to appear 2011.
- [7] M. Dos Santos, The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part II: the non-zero degree case, preprint.
- [8] A. Kachmar, Magnetic vortices for a Ginzburg-Landau type energy with discontinuous constraint. *ESAIM Control Optim. Calc. Var.* 16 (2010), no. 3, 545 - 580.
- [9] H. Aydi, A. Kachmar, Magnetic vortices for a Ginzburg-Landau type energy with discontinuous constraint. II. *Commun. Pure Appl. Anal.* 8 (2009), no. 3, 977 - 998.
- [10] L. Lassoued, P. Mironescu, Ginzburg-Landau type energy with discontinuous constraint. *J. Anal. Math.* 77 (1999), 1 - 26.
- [11] P. Pedregal, *Parametrized measures and variational principles*, Birkhauser (1997).
- [12] I. Ekeland; R. Temam, *Analyse convexe et problemes variationnels*. (French) Collection Etudes Mathematiques. Dunod; Gauthier-Villars, Paris-Brussels-Montreal, Que., 1974.

- [13] M. Valadier, Young measures, Methods of Nonconvex Analysis, Lecture Notes Math., Springer (1990) pp. 152 - 188.

5 Appendix

The derivation of the lower bound in Section 3 makes use of the concept of Young measures. Recall that a Young measure is a parametrized family of probability measures m_x associated to a family of functions $\phi_\varepsilon(x)$ ($\phi_\varepsilon : \Omega \rightarrow \mathbb{R}$) such that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} F(\phi_\varepsilon(x)) dx = \int_{\Omega} \int_{\mathbb{R}} F(\lambda) dm_x(\lambda) dx, \quad (45)$$

for every bounded continuous function $F(\lambda)$. It is known (see, e. g., [4], [11], [13]) that under some a priori bounds on the sequence of function $\phi_\varepsilon(x)$ there exists a family m_x such that (45).

In this work we consider a sequence of integer valued functions $D^\varepsilon(x)$. We construct a partition of unity $\mu_k(x)$ associated this sequence via (22)-(23), so that the corresponding Young measure m_x on \mathbb{R}^1 is the sum of δ -functions centered at integer points, $m_x(\lambda) = \sum \mu_k(x) \delta_k(\lambda)$. For fixed k the value $\mu_k(x)$ represents the probability to find a hole ω_j^ε with degree $d_j^\varepsilon = k$ in a small vicinity of the point x (i.e. $\mu_k(x)$ represents the ratio of holes with degree k in a small neighborhood of x to the total number of holes in this neighborhood).